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ОБ ОДНОМ ОБОБЩЕНИИ ЛОКАЛЬНЫХ ФОРМАЦИЙ

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ON ONE GENERALIZATION OF THE LOCAL FORMATIONS

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Все рассматриваемые в работе группы предполагаются конечными. Пусть $\sigma=\{\sigma_i \mid i\in I\}$ – некоторое разбиение множества всех простых чисел $\mathbb{P}$. Натуральные числа *n* и *m* называются σ*-взаимно простыми*, если для всякого такого σ_i, что $\sigma_i\cap\pi(n)\neq\varnothing$, мы имеем $\sigma_i\cap\pi(m)=\varnothing$. Пусть $t>1$ – натуральное число и пусть $\mathfrak{F}$ – класс групп. Тогда мы говорим, что $\mathfrak{F}$ является: (i) $\mathcal{S}_\sigma^t$*-замкнутым* (соответственно *слабо* $\mathcal{S}_\sigma^t$*-замкнутым*), если $\mathfrak{F}$ содержит всякую конечную группу *G*, удовлетворяющую следующим условиям: (1) *G* содержит такие подгруппы $A_1,\ldots,A_t\in\mathfrak{F}$, что $G=A_iA_j$ для всех $i\neq j$; (2) индексы $|G:N_G(A_1)|,\ldots,|G:N_G(A_t)|$ (соответственно индексы $|G:A_1|,\ldots,|G:A_{t-1}|$, $|G:N_G(A_t)|$) попарно σ-взаимно просты; (ii) $\mathcal{M}_\sigma^t$*-замкнутым* (соответственно *слабо* $\mathcal{M}_\sigma^t$*-замкнутым*), если $\mathfrak{F}$ содержит всякую конечную группу *G*, удовлетворяющую следующим условиям: (1) *G* содержит такие модулярные подгруппы $A_1,\ldots,A_t\in\mathfrak{F}$, что $G=A_iA_j$ для всех $i\neq j$; (2) индексы $|G:N_G(A_1)|,\ldots,|G:N_G(A_t)|$ (соответственно индексы $|G:A_1|,\ldots,|G:A_{t-1}|,|G:N_G(A_t)|$) попарно σ-взаимно просты. В работе изучаются свойства и приложения (слабо) $\mathcal{S}_\sigma^t$-замкнутых и (слабо) $\mathcal{M}_\sigma^t$-замкнутых классов конечных групп.

Ключевые слова: *конечная группа*, *формационная* σ*-функция*, σ*-локальная формация*, (*слабо*) S_σ^t*-замкнутый класс групп*, (*слабо*) M_σ^t *-замкнутый класс групп.*

Throughout this paper, all groups are finite. Let $\sigma=\{\sigma_i \mid i\in I\}$ be some partition of the set of all primes $\mathbb{P}$. The natural numbers *n* and *m* are called σ*-coprime* if for every σ_i such that $\sigma_i\cap\pi(n)\neq\varnothing$ we have $\sigma_i\cap\pi(m)=\varnothing$. Let $t>1$ be a natural number and let $\mathfrak{F}$ be a class of groups. Then we say that $\mathfrak{F}$ is: (i) $\mathcal{S}_\sigma^t$*-closed* (respectively *weakly* $\mathcal{S}_\sigma^t$*-closed*) provided $\mathfrak{F}$ contains each finite group *G* which satisfies the following conditions: (1) *G* has subgroups $A_1,\ldots,A_t\in\mathfrak{F}$ such that $G=A_iA_j$ for all $i\neq j$; (2) The indices $|G:N_G(A_1)|,\ldots,|G:N_G(A_t)|$ (respectively the indices $|G:A_1|,\ldots,|G:A_{t-1}|$, $|G:N_G(A_t)|$) are pairwise σ-coprime); (ii) $\mathcal{M}_\sigma^t$*-closed* (respectively *weakly* $\mathcal{M}_\sigma^t$*-closed*) provided $\mathfrak{F}$ contains each finite group *G* which satisfies the following conditions: (1) *G* has modular subgroups $A_1,\ldots,A_t\in\mathfrak{F}$ such that $G=A_iA_j$ for all $i\neq j$; (2) The indices $|G:N_G(A_1)|,\ldots,|G:N_G(A_t)|$ (respectively the indices $|G:A_1|,\ldots,|G:A_{t-1}|,|G:N_G(A_t)|$) are pairwise σ-coprime. In this paper, we study properties and applications of (weakly) $\mathcal{S}_\sigma^t$-closed and (weakly) $\mathcal{M}_\sigma^t$-closed classes of finite groups.

Keywords: *finite group*, *formation* σ*-function*, σ*-local formation*, (*weakly*) $\mathcal{S}_\sigma^t$*-closed class of groups*, (*weakly*) $\mathcal{M}_\sigma^t$*-closed class of groups.*

1 Preliminaries

We use the terminology in [1]–[3]. Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi=\{p_1,\ldots,p_n\}\subseteq\mathbb{P}$ and $\pi'=\mathbb{P}\setminus\pi$. If *n* is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing *n*; as usual, $\pi(G)=\pi(|G|)$, the set of all primes dividing the order of *G*.

A subgroup *M* of *G* is called *modular* if *M* is a modular element (in the sense of Kurosh [4, p. 43]) of the lattice $\mathcal{L}(G)$ of all subgroups of G, that is,

(i) $\langle X, M\cap Z\rangle=\langle X,M\rangle\cap Z$ for all $X\leq G$, $Z\leq G$ such that $X\leq Z$, and

(ii) $\langle M, Y\cap Z\rangle=\langle M,Y\rangle\cap Z$ for all $Y\leq G$, $Z\leq G$ such that $M\leq Z$.

In what follows, σ is some partition of $\mathbb{P}$, that is, $\sigma=\{\sigma_i \mid i\in I\}$, where $\mathbb{P}=\bigcup_{i\in I}\sigma_i$ and $\sigma_i\cap\sigma_j=\varnothing$ for all $i\neq j$; $\Pi\subseteq\sigma$ and $\Pi'=\sigma\setminus\Pi$.

By the analogy with the notation $\pi(n)$, we write $\sigma(n)$ [5] to denote the set

$$\{\sigma_i \mid \sigma_i\cap\pi(n)\neq\varnothing\};\ \sigma(G)=\sigma(|G|)$$

and $\sigma(\mathfrak{M}) = \bigcup_{G\in\mathfrak{M}} \sigma(G)$. The natural number n and m are called *σ-coprime* if $\sigma(n) \cap \sigma(m) = \varnothing$.

Recall that a class of groups $\mathfrak{F}$ is called a *formation* if: (i) $G/N \in \mathfrak{F}$ whenever $G \in \mathfrak{F}$, and (ii) $G/(N\cap R) \in \mathfrak{F}$ whenever $G/N \in \mathfrak{F}$ and $G/R \in \mathfrak{F}$.

Following [6], we call any function f of the form

$$f : \sigma \to \{\text{formations of groups}\}$$

a formation σ-function, and we put

$LF_\sigma(f) = (G \mid G/O_{\sigma_i',\sigma_i}(G) \in f(\sigma_i)$ for all $\sigma_i \in \sigma(G))$.

***Definition* 1.1** [6]. (i) If for some formation σ-function f we have $\mathfrak{F} = LF_\sigma(f)$, then we say that the class $\mathfrak{F}$ is *σ-local* and f is a *σ-local definition of* $\mathfrak{F}$.

(ii) We suppose that every formation is *0-multiply σ-local*; for $n > 0$, we say that the formation $\mathfrak{F}$ is *n-multiply σ-local* provided either $\mathfrak{F} = (1)$ is the class of all identity groups or $\mathfrak{F} = LF_\sigma(f)$, where $f(\sigma_i)$ is $(n-1)$-multiply local for all $\sigma_i \in \sigma(\mathfrak{F})$.

(iii) The formation $\mathfrak{F}$ is said to be *totally σ-local* provided $\mathfrak{F}$ is *n*-multiply σ-local for all $n \in \mathbb{N}$.

In this paper, we discuss applications of the theory of σ-local formations [6] in the study of factorizations of groups.

***Remark* 1.2.** (i) In view of [7, IV, 3.2], in the classical case when $\sigma = \{\{2\},\{3\},\ldots\}$, a formation σ-function and a σ-local formation are, respectively, a formation function and a local formation in the usual sense [7, IV, Definition 3.1].

(ii) By definition, the class of all identity groups (1) is totally σ-local and it is contained in $LF_\sigma(f)$. Moreover, $(1) = LF_\sigma(h)$, where $h(\sigma_i) = \varnothing$ for all σ_i.

If $t > 1$ is a natural number, then the class of groups $\mathfrak{F}$ is said to be *Σ_t-closed* (Shemetkov [8]) provided $\mathfrak{F}$ contains all groups G which satisfy the following condition: G has subgroups $A_1,\ldots,A_t \in \mathfrak{F}$ whose indices $|G:A_1|,\ldots,|G:A_t|$ are pairwise coprime. The theory of Σ_t-closed classes of soluble groups and various its applications were considered by Otto-Uwe-Kramer in [9] (see also [8, Chapter 1]).

In this paper we consider the following generalizations of this concept.

***Definition* 1.3** [10]. Let $t > 1$ be a natural number and let $\mathfrak{F}$ be a class of groups. Then we say that $\mathfrak{F}$ is *$\mathcal{S}_\sigma^t$-closed* (respectively *$\mathcal{M}_\sigma^t$-closed*) if $\mathfrak{F}$ contains every group G for which the following conditions hold:

(1) G has subgroups (respectively modular subgroups) $A_1,\ldots,A_t \in \mathfrak{F}$ such that $G = A_iA_j$ for all $i \neq j$;

(2) The indices $|G:N_G(A_1)|,\ldots,|G:N_G(A_t)|$ are pairwise σ-coprime.

***Definition* 1.4** [10]. Let $t > 1$ be a natural number and let $\mathfrak{F}$ be a class of groups. Then we say that $\mathfrak{F}$ is *weakly $\mathcal{S}_\sigma^t$-closed* (respectively weakly *$\mathcal{M}_\sigma^t$-closed*) if $\mathfrak{F}$ contains every group G for which the following conditions hold:

(1) G has subgroups (respectively modular subgroups) $A_1,\ldots,A_t \in \mathfrak{F}$ such that $G = A_iA_j$ for all $i \neq j$;

(2) The indices $|G:A_1|,\ldots,|G:A_{t-1}|,|G:N_G(A_t)|$ are pairwise σ-coprime.

In the case when $\sigma = \sigma^1 = \{\{2\},\{3\},\ldots\}$, the symbol σ^1 will be omitted. Hence the class $\mathfrak{F}$ is $\mathcal{S}^t$-closed (respectively is *weakly $\mathcal{S}^t$-closed*) if it is $\mathcal{S}_{\sigma^1}^t$-closed (respectively weakly $\mathcal{S}_{\sigma^1}^t$-closed); the class $\mathfrak{F}$ is $\mathcal{M}^t$-closed (respectively is weakly $\mathcal{M}^t$-closed) if it is $\mathcal{M}_{\sigma^1}^t$-closed (respectively weakly $\mathcal{M}_{\sigma^1}^t$-closed).

2 Main Results

Recall that G is said to be: *σ-primary* [11] if G is a σ_i-group for some i; *σ-decomposable* (Shemetkov [8]) or *σ-nilpotent* (Guo and Skiba [12]) if $G = G_1\times\ldots\times G_n$ for some σ-primary groups $G_1,\ldots,G_n$; *σ-soluble* [11] if every chief factor of G is σ-primary.

We say that G is *meta-σ-nilpotent* if G has a normal subgroup N such that N and G/N are σ-nilpotent.

Our main result is the following

***Theorem* 2.1** [10]. *The following statements hold*:

(i) *The formation of all meta-σ-nilpotent groups $\mathfrak{N}_\sigma^2$ is $\mathcal{S}_\sigma^4$-closed and $\mathcal{M}_\sigma^3$-closed.*

(ii) *Every σ-local formation of meta-σ-nilpotent groups is weakly $\mathcal{S}_\sigma^4$-closed.*

(iii) *Every 2-multiply σ-local formation of meta-σ-nilpotent groups is weakly $\mathcal{M}_\sigma^3$-closed.*

Before continuing, consider some corollaries of Theorem 2.1. First note that in view of Remark 1.2 (i), we get from this result the following

Corollary 2.2. *The following statements hold*:

(i) *The formation of all meta-nilpotent groups $\mathfrak{N}^2$ is $\mathcal{S}^4$-closed and $\mathcal{M}^3$-closed.*

(ii) *Every local formation of meta-nilpotent groups is weakly $\mathcal{S}^4$-closed.*

(iii) *Every 2-multiply local formation of meta-nilpotent groups is weakly $\mathcal{M}^3$-closed.*

It is clear that every Σ_t-closed class is also weakly $\mathcal{S}^4$-closed. Hence we get from Corollary 2.2 the following

***Corollary* 2.3** (Otto-Uwe-Kramer [9]). *Every local formation of meta-nilpotent groups is Σ_4-closed.*

From Corollary 2.2 we get also the following two new results

***Corollary* 2.4.** *Suppose that G has supersoluble subgroups A_1, A_2, A_3, A_4 such that $G = A_iA_j$ for all $i \neq j$, and the indices*

$$|G : A_1|, |G : A_2|, |G : A_3|, |G : N_G(A_4)|$$

are pairwise coprime. Then G is supersoluble.

***Corollary* 2.5.** *Suppose that G has subgroups A_1, A_2, A_3, A_4 such that $G = A_iA_j$ and A_i' is nilpotent for all $i \neq j$, and the indices*

$$|G : A_1|, |G : A_2|, |G : A_3|, |G : N_G(A_4)|$$

are pairwise coprime. Then G' is nilpotent.

From Corollary 2.4 we get

***Corollary* 2.6** (Doerk [13]). *Suppose that G has supersoluble subgroups A_1, A_2, A_3, A_4 such that $G = A_iA_j$ for all $i \neq j$, and the indices*

$$|G : A_1|, |G : A_2|, |G : A_3|, |G : A_4|$$

are pairwise coprime. Then G is supersoluble.

From Corollary 2.5 we get

***Corollary* 2.7** (Friesen [14]). *Suppose that G has subgroups A_1, A_2, A_3, A_4 such that $G = A_iA_j$ and A_i' is nilpotent for all $i \neq j$, and the indices*

$$|G : A_1|, |G : A_2|, |G : A_3|, |G : A_4|$$

are pairwise coprime. Then G' is nilpotent.

Let $\pi = \{p_1, p_2, ..., p_n\}$. Then G is said to be π-*nilpotent* (respectively π-*decomposable*) if $G = O_{p_1}(G) \times \cdots \times O_{p_n}(G) \times O_{\pi'}(G)$ (if, respectively, if $G = O_{\pi}(G) \times O_{\pi'}(G)$).

In fact, in the theory of the π-soluble groups we deal with the partition $\sigma = \{\{p_1\}, ..., \{p_n\}, \pi'\}$ of $\mathbb{P}$. In this case we get from Theorem 2.1 the following

***Corollary* 2.8.** *Suppose that G has meta-π-nilpotent subgroups A_1, A_2, A_3, A_4 such that $G = A_iA_j$ for all $i \neq j$, and the indices*

$$|G : N_G(A_1)|, ..., |G : N_G(A_4)|$$

are pairwise coprime and also $|G : N_G(A_i)|$ is either a π-number or a π'-number for all i. Then G is meta-π-nilpotent.

If for a subgroup A of G we have $\sigma(|A|) \subseteq \Pi$ and $\sigma(|G : A|) \subseteq \Pi'$, then A is said to be a *Hall Π-subgroup* [5] of G. We say that G is Π-*closed* if G has a normal Hall Π-subgroup.

The proof of Theorem 2.1 consists of many steps and the next three basis theorems and the proposition are the main stages of it.

***Theorem* 2.9** [10]. (i) *The class of all σ-soluble Π-closed groups $\mathfrak{F}$ is $\mathcal{S}_{\sigma}^{3}$-closed.*

(ii) *The class of all σ-nilpotent groups $\mathfrak{N}_{\sigma}$ is $\mathcal{S}_{\sigma}^{3}$-closed.*

(iii) *Every formation of σ-nilpotent groups $\mathfrak{M}$ is weakly $\mathcal{S}_{\sigma}^{3}$-closed.*

In the case when $\sigma = \sigma^1 = \{\{2\}, \{3\}, ...\}$ we get from Theorem 2.9 the following results.

***Corollary* 2.10.** *Suppose that*

$$G = A_1A_2 = A_2A_3 = A_1A_3,$$

where A_1, A_2 and A_3 are soluble subgroups of G. If the three indices $|G : N_G(A_1)|$, $|G : N_G(A_2)|$, $|G : N_G(A_3)|$ are pairwise coprime, then G is soluble.

***Corollary* 2.11** (Wielandt [15]). *If G has three soluble subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime, then G is itself soluble.*

***Corollary* 2.12.** *Suppose that*

$$G = A_1A_2 = A_2A_3 = A_1A_3,$$

where A_1, A_2 and A_3 are abelian subgroups of G. If the three indices $|G : A_1|$, $|G : A_2|$, $|G : N_G(A_3)|$ are pairwise coprime, then G is abelian.

***Corollary* 2.13.** *Suppose that*

$$G = A_1A_2 = A_2A_3 = A_1A_3,$$

where A_1, A_2 and A_3 are nilpotent subgroups of G. If the three indices $|G : A_1|$, $|G : A_2|$, $|G : N_G(A_3)|$ are pairwise coprime, then G is nilpotent.

***Corollary* 2.14** (Kegel [16]). *Suppose that $G = A_1A_2 = A_2A_3 = A_1A_3$, where A_1, A_2 and A_3 are nilpotent subgroups of G. If the three indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime, then G is nilpotent.*

***Corollary* 2.15** (Doerk [13]). *Suppose that $G = A_1A_2 = A_2A_3 = A_1A_3$, where A_1, A_2 and A_3 are abelian subgroups of G. If the three indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime, then G is abelian.*

In the case when $\sigma = \{\{p_1\}, ..., \{p_n\}, \pi'\}$ we get from Theorem 2.9 the following results.

***Corollary* 2.16.** *Suppose that G has π-nilpotent subgroups A_1, A_2, A_3 such that $G = A_iA_j$ for all $i \neq j$, and the indices*

$$|G : N_G(A_1)|, |G : N_G(A_2)|, |G : N_G(A_3)|$$

are pairwise coprime and also $|G : N_G(A_i)|$ is either a π-number or a π'-number for all i. Then G is π-nilpotent.

***Corollary* 2.17.** *Suppose that G has π-nilpotent subgroups A_1, A_2, A_3 such that $G = A_iA_j$ for all $i \neq j$, and the indices*

$$|G : A_1|, |G : A_2|, |G : N_G(A_3)|$$

are pairwise coprime and also every of the indices $|G : A_1|, |G : A_2|, |G : N_G(A_3)|$ is either a π-number or a π'-number. If the Sylow subgroups of A_i are abelian for all i, then the Sylow subgroups of G are also abelian.

***Theorem* 2.18** [10]. *If $\mathfrak{F} = LF_{\sigma}(f)$ is a σ-local formation of σ-soluble groups, where for every i the formation $f(\sigma_i)$ is* (*weakly*) $\mathcal{S}_{\sigma}^{t}$*-closed, then the class $\mathfrak{F}$ is* (*weakly*) $\mathcal{S}_{\sigma}^{t+2}$*-closed.*

***Proposition* 2.19** [6]. *Every σ-local formation of σ-soluble groups $\mathfrak{F}$ possesses a unique σ-local definition F such that for every σ-local definition f of $\mathfrak{F}$ and for every $\sigma_i \in \sigma$ the following inclusions hold*:

$$f(\sigma_i) \cap \mathfrak{F} \subseteq F(\sigma_i) = \mathfrak{S}_{\sigma_i} F(\sigma_i) \subseteq \mathfrak{F}.$$

In this proposition the symbol $\mathfrak{S}_{\sigma_i}$ denotes the class of all σ-soluble σ_i-soluble groups, $\mathfrak{S}_{\sigma_i} F(\sigma_i)$ denotes the class of all σ-soluble groups G such that for some normal subgroup N of G we have $G / N \in F(\sigma_i)$.

We call the function F in Proposition 2.19 the *canonical σ-local definition* of $\mathfrak{F}$.

Our fourth basic result is the following

***Theorem* 2.20** [10]. *If $\mathfrak{F} = LF_{\sigma}(F)$ is a σ-local formation of σ-soluble groups*, *where F is the canonical σ-local definition of $\mathfrak{F}$ and for every i the formation $F(\sigma_i)$ is is* (*weakly*) $\mathcal{S}_{\sigma}^{t}$*-closed, then the formation $\mathfrak{F}$ is* (*weakly*) $\mathcal{S}_{\sigma}^{t+1}$*-closed.*

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